## 4.1: First Order Systems and Applications

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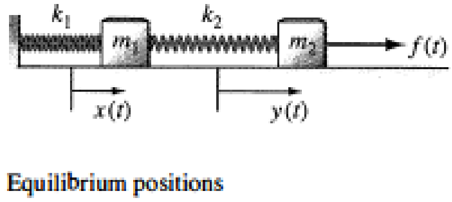
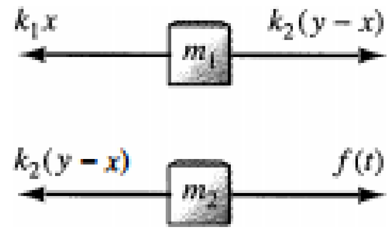
**Example 1.**




Consider the system of two masses and two springs shown above left, with an external force $f(t)$ acting on the right mass $m_2$. Applying Newton's law of motion to the two "free body diagrams" shown top right, we obtain the system

$$\begin{aligned} m_1x'' &= -k_1x + k_2(y-x) \\ m_2y'' &= -k_2(y-x) + f(t). \end{aligned}$$

If, for instance, $m_1 = 2, m_2 = 1, k_1 = 4$ and $f(t) = 40\sin 3t$ then we arrive at

$$\begin{aligned} 2x'' &= -6x + 2y \\ y'' &= 2x - 2y + 40\sin 3t. \end{aligned}$$

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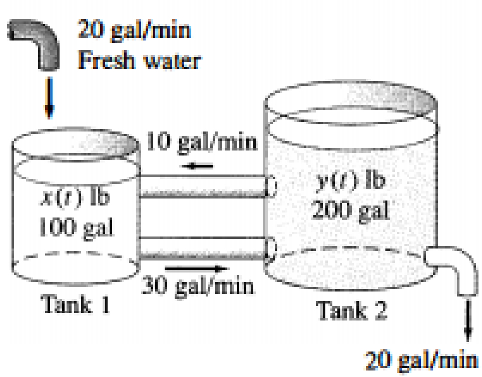
**Example 2.**




Consider two brine tanks (shown left). Tank 1 contains $x(t)$ pounds of salt in 100 gal of brine and tank 2 contains $y(t)$ pounds of salt in 200 gal of brine. Everything is kept uniform by stirring as tank 1 receives 20 gal/min of fresh water and tank 2 flows out at 20 gal/min. Computing the rate of change of salt in each tank, we arrive at

$$\begin{aligned} x' &= -30 \cdot \frac{x}{100} + 10 \cdot \frac{y}{200} \\ y' &= 30 \cdot \frac{x}{100} - 10 \cdot \frac{y}{200} - 20 \cdot \frac{y}{200}. \end{aligned}$$

In this section we wish to only consider first-order systems. In order to do this, we will change higher-order systems into first-order systems.

**Example 3.** Rewrite the third-order system

$$x^{(3)} + 3x'' + 2x' - 5x = \sin 2t$$

as an equivalent first-order system of equations.

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**Example 4.** Rewrite the second-order system

$$\begin{aligned} 2x'' &= -6x + 2y \\ y'' &= 2x - 2y + 40 \sin 3t \end{aligned}$$

as an equivalent first-order system of equations.

**Example 5.** Solve the two-dimensional system

$$x' = -2y, \quad y' = \tfrac{1}{2}x.$$

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**Example 6.** Find the general solution of the system

$$x' = y, \quad y' = 2x + y.$$

**Example 7.** Solve the initial value problem

$$\begin{aligned} x' &= -y, \\ y' &= (1.01)x - (0.2)y, \\ x(0) = 0, &\quad y(0) = 1. \end{aligned}$$

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**Theorem 1.** (Existence and Uniqueness for Linear Systems)
Suppose that the functions $p_{11}, p_{12}, \ldots, p_{nn}$ and the functions $f_1, \ldots, f_n$ are continuous on the open interval $I$ containing $a$. Then, given the $n$ numbers $b_1, \ldots, b_n$, the linear system has a unique solution on the entire interval $I$ that satisfies the $n$ initial conditions

$$x_1(a) = b_1, \quad \cdots, \quad x_n(a) = b_n.$$

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**Homework.** 1-7, 17-25 (odd)